Math 2010B Tutorial 12.

HWS $\Longrightarrow$ $\begin{cases} \text{yqhuang@math.cuhk.edu.hk} & \text{Yiqi Hung} \quad \text{tutor} \\ \text{chcheung@math.cuhk.edu.hk.} & \text{CHEUNG Chin Ho} \quad \text{tutor} \end{cases}$

Outline:

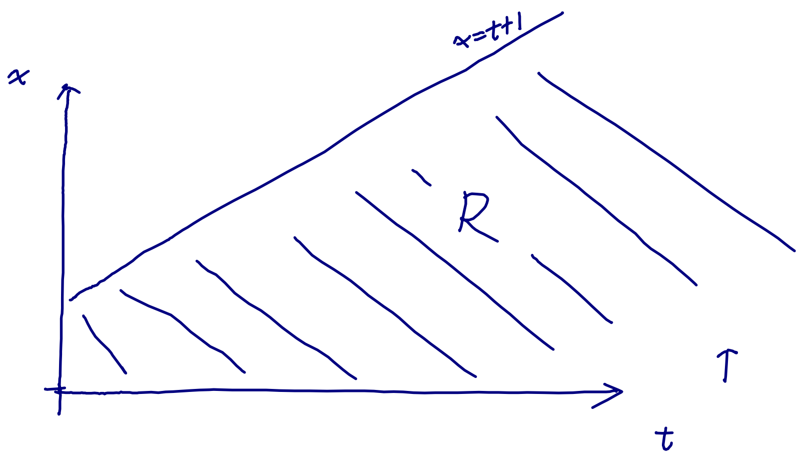
- Optimization problem
- Second Derivative Test

e.g. Let $f(t,x) = (x^2 - 2x + t)\, e^{-t}$

Find the global maximum of f (if it exists) on the region

$C \subset R = \{ (t,x) \in \mathbb{R}^2 : t \geq 0,\ 0 \leq x \leq t+1 \}$

Sol: Observe the nature of the region on which f is to be maximized



R is closed, but not bounded.

Strategy: Find

① a closed and bounded subset C of R; and

② a suitable pt $(t_0, x_0) \in C$ (for comparison)

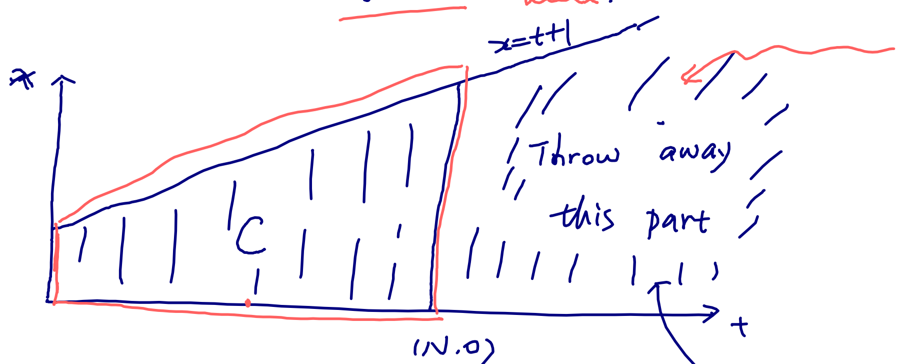
s.t. $f(t,x) \boxed{\le} f(t_0, x_0)$ $\forall (t,x) \in R \setminus C$, if possible

Claim: $\exists N > 0$ s.t. $\forall (x,y) \in R$. w/ $t \geq N$

$$f(t,x) \le f(3,0) = e^{-3}.$$

$f(t,x) = \underbrace{(x^2 - 2x + t)}_{\Delta} \underbrace{e^{-t}}_{\Delta}$

$\lim_{t \to \infty} e^{-t} = 0$



$f(t,x) \le f(3,0)$

Then $C := \{(t,x) \in R : t \le N\}$ is closed and bounded.

By EVT, f attains a max on C, which is also a maxi on R

by the above claim.

- proof of claim:

$\forall (t,x) \in R$,

$$|f(t,x)| = |(x-1)^2 + t - 1|\, e^{-t}$$

$$\le ((x-1)^2 + t + 1)\, e^{-t} \qquad \triangle \text{ inequ}$$

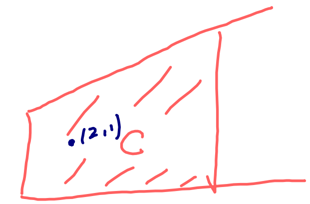
$$\le (t^2 + t + 1)\, e^{-t} \qquad (x \le t+1)$$

$$\boxed{\lim_{t \to +\infty} (t^2+t+1)e^{-t} = \lim_{t \to +\infty} \frac{t^2+t+1}{e^t} \rightsquigarrow 2t \rightsquigarrow 2 \to 0 = 0}$$ (L'Hopital's Rule on $\frac{\infty}{\infty}$ form)

Note $f(3,0) = e^{-3} > 0$ $\therefore$ $\exists N > 0$ (in fact $N > 3$) s.t

$\forall (t,x) \in R$ w/ $t \ge N$, $f(t,x) \le |f(t,x)| \le (t^2 + t + 1)e^{-t} \le f(3,0)$.

Now C is compact $\qquad f(x,t) = (x^2 - 2t + t)\, e^{-t}$



- Find critical points of f in the interior of C.

Let $g(x) = x^2 - 2x$ Then $g'(x) = \underset{\substack{\downarrow \\ x=1}}{2(x-1)}$; $g''(x) = 2$

$$f_t = -(t - 1 + g(x))\, e^{-t} \quad ; \quad f_x = g'(x)\, e^{-t}$$

Solving $(f_t, f_x) = (0,0)$, we get $(t,x) = (2,1)$.

$(2,1)$ is the unique critical pt of f in the interior of C.

Q: Does f take local maxi at $(2,1)$?

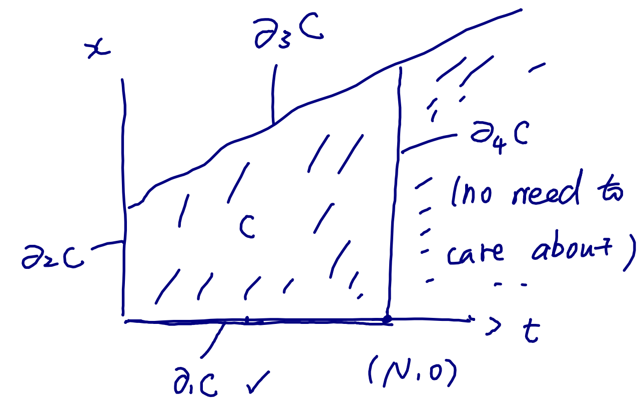
$$f_{xx} = \underbrace{g''(x)}_{=2}\, e^{-t} = 2 \cdot e^{-t} \qquad f_{tt} = (t - 2 + g(x))\, e^{-t}$$

$$f_{xt} = f_{tx} = -g'(x)\, e^{-t}$$

$$f_{xx}(2,1)\, f_{tt}(2,1) - f_{xt}^2(2,1) = 2e^{-2}(-e^{-2}) - 0^2 < 0$$

$\therefore$ $(2,1)$ is a saddle pt of f (by Second Derivative Test)

(We do not care about saddle pts)



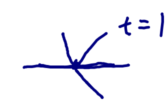
- Find max of f on ∂C

For $\partial_1 C$

Let $h_1(t) = f(t, \overset{x=0}{0}) = t \cdot e^{-t}$

$h_1'(t) = (1-t)e^{-t}$ $\rightsquigarrow$ $t=1$ critical point.

$h_1''(t) = (t-2)e^{-t}$ $\qquad h_1''(1) = -e^{-1} < 0$



for 1 variable.

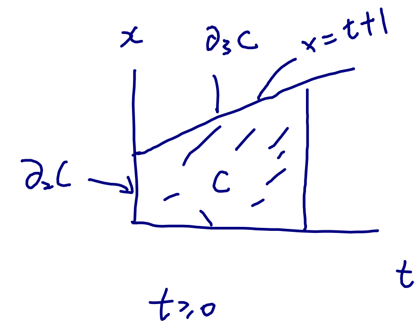
$h_1'(t) > 0$ for $t < 1$

$h_1'(t) < 0$ for $t > 1$

on $(0,N)$, (i.e $t \in (0,N)$)

h_1 has a local max at 1 w/ $f(1,0) = h_1(1) = e^{-1}$

For $\partial_2 C$



Let $h_2(x) = f(0, x) = g(x)$

on $(0,1)$ $h_2(g)$ is strictly decreasing.

For $\partial_3 C$

$$h_3(t) = f(t,x)\Big|_{x=t+1} = f(t, t+1) = (t^2+t-1)e^{-t}$$

$$h_3'(t) = -(t^2-t-2)e^{-t} = -(t-2)(t+1)e^{-t}$$

$$h_3''(t) = (t^2-3t-1)e^{-t}$$

Solving $h_3'(t) = 0 \Rightarrow \underline{t=2}$ or $t=-1$ (rejected)

$$h_3''(t) = -3e^{-2} < 0$$

on $(0,N)$, (i.e $t \in (0,N)$) h_3 has a local max at $t=2 \Rightarrow x = t+1 = 3$

w/ $h_3(2) = f(2,3) = 5e^{-2}$ and no other critical pt.

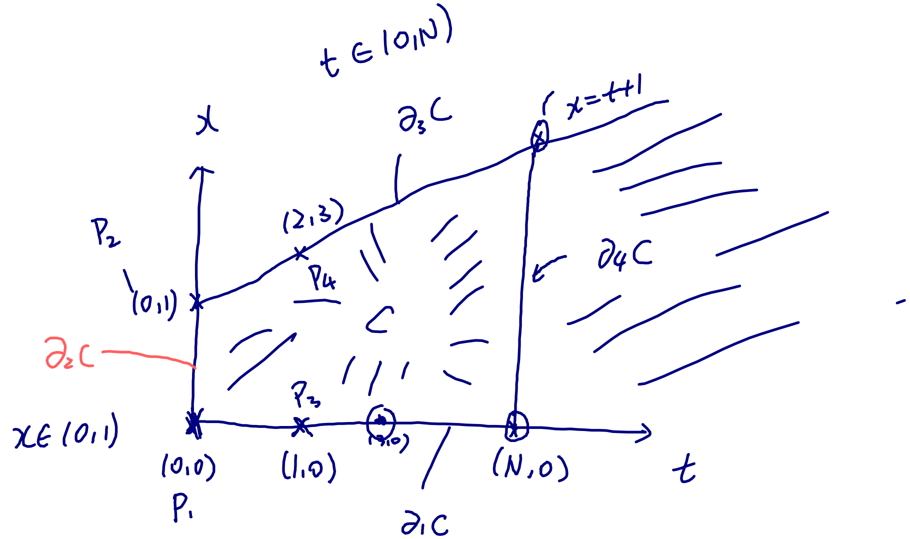
Comparing the possible pts

$f(0,0) = 0$

$f(0,1) = -1$

$\underline{f(1,0) = e^{-1}}$

$\underline{f(2,3) = 5e^{-2}}$



$\because$ $e < 5$, $\quad f(1,0) = e^{-1} < 5 \cdot e^{-2} = f(2,3)$

Conclusion:

f has a global max at $(2,3)$ on R w $f(2,3) = 5 \cdot e^{-2}$

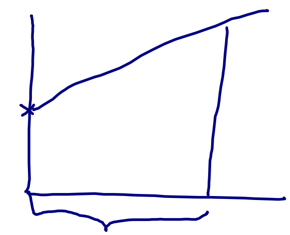
Excise : Let $f(x,y) = \max\{g(x,y),\ g(x,y+1)\}$

where $g(x,y) = x^2 + xy + y^2$

Find the global maximum of (if it exists) on $\mathbb{R}^2$

---

why should consider $f(0,0)$ $f(0,1)$



for $\partial_1 C$

$h_1(t) = te^{-t}$ $t \in [0, N]$

$h_1(0)$ & $h_1(N)$

$\parallel$

$f(0,0)$

for $t \in (0,N)$ we compute local maximal.

so we also need to compare with boundary point.